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Asymptotic level spacing of the Laguerre ensemble: a Coulomb fluid approach

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Abstract. We determine the asymptotic level spacing distribution for the Laguerre ensemble in a single-scaled interval, $(0, s)$, containing no levels, $E_\beta(0, s)$, via Dyson's Coulomb-fluid approach. For the $\alpha = 0$ unitary Laguerre ensemble, we recover the exact spacing distribution found by both Edelman and Forrester, while for $\alpha \neq 0$, the leading terms of $E_2(0, s)$, found by Tracy and Widom, are reproduced without the use of the Bessel kernel and the associated Painlevé transcendent. In the same approximation, the next leading term, due to a 'finite-temperature' perturbation ($\beta \neq 2$), is found.

1. Introduction

The probability that there are no levels in a scaled interval $(-t, t)$ (where t is measured with respect to the averaged spacing), $E(0, t)$, in a long stack of energy levels of heavy nuclei was given by a conjecture of Wigner [1] and was well supported by experimental data [2] for systems with time-reversal symmetry. In a series of seminal papers, Dyson [3] introduced a new class of random matrix ensembles and determined in the continuum approximation (expected to be valid when number of levels, N, is very large) that $\ln E_\beta(0, t) \sim -(\pi^2/4)\beta t^2 - (1 - \beta/2)\pi t$, using the methods of classical electrostatics, potential theory and thermodynamics, for ensembles with unitary ($\beta = 2$), orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) symmetries. A term of $O(\ln t)$ and a constant missed in the continuum approximation was later discovered by Widom, des Cloizeaux and Mehta, and Dyson [4–8]†.

Recently, in a remarkable paper, Tracy and Widom [9] showed, in the single-interval case, that the logarithmic derivative of the Fredholm determinant of the Bessel Kernel—which arises in the scaling limit (with respect to the hard edge density, see [9, 10] of the unitary Laguerre ensemble—satisfies a Painlevé V equation, from which the asymptotic level spacing distribution can be computed exactly, amongst other quantities of interest from the random-matrix point of view‡.

In this paper we shall employ the continuum approximation of Dyson to calculate the level-spacing distribution for the Laguerre ensemble, with $\beta = 2$. For $\beta \neq 2$, the spacing distribution can be found by a perturbative calculation due to Dyson [3].

† In an application of the continuum approximation, the quantity $r_\beta(n, t) := E_\beta(n, t)/E_\beta(0, t)$ was calculated, where $E_\beta(n, t)$ is the probability that there are exactly n levels contained in the interval $(-t, t)$, with $1 \ll n \ll t$. See [7]. $r_\beta(n, t)$ is also computed in [8]. As will be seen later, the computation of $E_\beta(0, t)$, involves subtraction of two large terms which are functions of N and the continuum approximation is not expected to have sufficiently fine resolution to determine the constant.

‡ The Painlevé V in [9] is reducible to a Painlevé III.

From the Brownian-motion model, in the 'hydrodynamical' approximation, Dyson [11] derived an equation satisfied by the non-equilibrium level density $\sigma(x, \tau)$:

$$\frac{\partial}{\partial \tau}\sigma(x,\tau) = \frac{\partial}{\partial x}\left(\beta\sigma(x,\tau)\frac{\partial \Psi}{\partial x}\right) \qquad \tau > 0 \tag{1}$$

with

$$\Psi(x,\tau) = \frac{1}{\beta}u(x) - \int \mathrm{d}y\, \sigma(y,\tau)\ln|x-y| + \left(\frac{1}{\beta} - \frac{1}{2}\right)\ln[\sigma(x,\tau)] \tag{2}$$

where the fictitious time τ pulls the levels towards the observed level density, $r(x)$, generated by the imposed potential $u(x)/\beta = \int \mathrm{d}y\, r(x)\ln|x-y|$, (as $\tau \to \infty$), that holds the Coulomb fluid together. The stationary solution, reached as $\tau \to \infty$, for the level density satisfies a Hückel-like self-consistent equation,

$$u(x) - \beta\int \mathrm{d}y\, \sigma(y)\ln|x-y| + \left(1 - \frac{\beta}{2}\right)\ln[\sigma(x)] = A = \text{constant} \tag{3}$$

with effective temperatute $T_e \propto (1 - \beta/2)$. This, in turn, may be derived from the following variational principle: $\min_{\sigma,\mu} F[\sigma, \mu]$, with

$$F[\sigma,\mu] = \beta\int \mathrm{d}x\, \Psi(x)\sigma(x) - \mu\left(\int \mathrm{d}x\, \sigma(x) - N\right) \tag{4}$$

and $A = -\mu + (1 - \beta/2)$, where μ is the chemical potential. Therefore the free energy, F, at equilibrium, with exactly N levels contained in an interval I is

$$F[I] = \frac{1}{2}AN + \frac{1}{2}\int_I \mathrm{d}x\, u(x)\sigma(x) - \frac{1}{2}\left(1 - \frac{\beta}{2}\right)\int_I \mathrm{d}x\, \sigma(x)\ln[\sigma(x)] \tag{5}$$

subject to $\int_I \mathrm{d}x\sigma(x) = N$. For the Laguerre ensemble, $u(x) = x - \alpha\ln x$, $x \in (0, \infty)$, but as we are using the continuum approximation, an upper-band edge, $b \in (0, \infty)$ must be imposed on the level density to produce a finite number of levels. In the large-N limit, the level density is $\sigma(x) = (1/2\pi)\sqrt{(4N-x)/x}$, $x \in (0, 4N)$ for the unitary case [12, 13]. This is distinct from the Wigner semicircle law, where $u(x) = x^2$, $x \in (-\infty, +\infty)$.

2. Asymptotic spacing

The probability distribution that an interval $(0, a)$ contains no levels is, by definition, the ratio of the partition function where all N levels reside in the complement of $(0, a)$ i.e. (a, b) (where b is the upper band edge and $0 < a < b$) to that for which all N levels reside in the full interval i.e. $(0, b)$, and is [3]

$$\ln E_\beta(0,a) = -[F(a,b) - F(0,b)]. \tag{6}$$

In the continuum approximation, F is given by (5), where $\sigma(x)$ solves (3)†. From thermodynamic considerations, since $F(a, b)$ is the free energy in a 'constricted' region, we must have $F(a, b) > F(0, b)$. We now proceed to solve (3) for $\beta = 2$, in the interval (a, b). To simplify the mathematics, (3) is converted into a singular integral equation by taking a derivative with respect to x, and can be solved by a standard method [14]. The solution of this equation is

$$\sigma(x) = \frac{1}{\pi^2\beta}\sqrt{\frac{b-x}{x-a}}\int_a^b \frac{\mathrm{d}y}{y-x}\sqrt{\frac{y-a}{b-y}}\left(1 - \frac{\alpha}{y}\right) = \frac{1}{\pi\beta}\sqrt{\frac{b-x}{x-a}}\left(1 - \frac{\alpha}{x}\sqrt{\frac{a}{b}}\right) \tag{7}$$

† For a discussion on the continuum approximation, the reader is urged to consult the second paper of [3].

where $x \in (a, b)$ and to maintain positivity we demand that $\sqrt{ab} > \alpha$. A straightforward calculation supplies the normalization condition,

$$N = \frac{b-a}{2\beta} + \frac{\alpha}{\beta}\left[\sqrt{\frac{a}{b}} - 1\right]. \tag{8}$$

We have deliberately left β without setting it equal to 2 in (7) and (8). It is clear from the structure of (8) that N, the total number of levels, is almost exhausted by the first term, but as to be seen later, the second term cannot be discarded. As we are required to compare $F(a, b)$ with $F(0, b)$, where N is very large, to facilitate the computation we shall evaluate instead, $F(a, b)$ in the limits $a \ll b$, $a/(b-a) \ll 1$ and $b/(b-a) \sim O(1)$, thus by-passing an independent computation of $F(0, b)$. To evaluate $F(a, b)$ requires the determination of A and the 'interaction energy' $\frac{1}{2}\int_a^b \mathrm{d}x\sigma(x)u(x)$. In the limits stated we shall extract the *very large* terms, which are functions of N only and finite terms that are functions of Na only; any remaining terms are therefore negligible in the large-N limit. Note that the very large terms are then subtracted according to the definition of $E_\beta(0, a)$, leaving behind only those terms in Na.

For the constant A, we send $x \to b$ in (3), which gives

$$A = a - \frac{b-a}{2}\ln\left[\frac{b-a}{4\mathrm{e}}\right] - \alpha\ln b - \alpha\left[\sqrt{\frac{a}{b}} - 1\right]\ln(b-a) + \frac{\alpha}{\pi}\sqrt{\frac{a}{b}}I\left(\frac{a}{b-a}\right) \tag{9}$$

where

$$\begin{aligned} I(x) &:= \int_0^1 \mathrm{d}t\,\sqrt{\frac{1-t}{t}}\,\frac{\ln(1-t)}{t+x} = \frac{1}{x}\frac{\partial}{\partial\nu}\left[B\left(\tfrac{1}{2}, \nu\right)F\left(1, \tfrac{1}{2}, \tfrac{1}{2}+\nu, -\frac{1}{x}\right)\right]\Bigg|_{\nu=3/2} \\ &\sim -2\pi(1-\ln 2) + \pi\sqrt{x} + \cdots \qquad x \ll 1 \end{aligned} \tag{10}$$

where $F(a, b, c, x) \equiv {}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function.

As $N \to \infty$, but with Na finite, we find that

$$\tfrac{1}{2}NA \sim -N^2\ln(N/\mathrm{e}) - \tfrac{1}{2}\alpha\ln(16N/\mathrm{e}^4) + \tfrac{1}{2}Na - \tfrac{1}{2}\alpha\sqrt{Na} \tag{11}$$

where to capture all the terms in Na it is essential to use (8) for N. For the contribution due to the interaction, we have

$$\begin{aligned} \frac{1}{2}\int_a^b \mathrm{d}t\,\sigma(t)\,u(t) &= \frac{1}{2}\left[\frac{(b-a)}{2\beta}\right]^2 + \frac{1}{2}a\,\frac{(b-a)}{2\beta} - \frac{\alpha}{2}\frac{(b-a)}{2\beta}\left[\sqrt{\frac{a}{b}} - 1\right] \\ &\quad - \frac{\alpha}{\pi}\frac{(b-a)}{2\beta}h\left(\frac{a}{b-a}\right) - \frac{\alpha^2}{2\beta}\ln(b-a)\left[\sqrt{\frac{a}{b}} - 1\right] + \frac{\alpha^2}{2\beta\pi}\sqrt{\frac{a}{b}}\,g\left(\frac{a}{b-a}\right) \end{aligned} \tag{12}$$

where

$$h(x) := \int_0^1 \mathrm{d}t\,\sqrt{\frac{1-t}{t}}\,\ln(t+x) \sim -\frac{\pi}{2}\ln(4\mathrm{e}) - \pi\,x + 2\pi\sqrt{x} + \cdots \qquad x \ll 1 \tag{13}$$

and

$$\begin{aligned} g(x) &:= \int_0^1 \frac{\mathrm{d}t}{t+x}\sqrt{\frac{1-t}{t}}\ln(t+x) = -\frac{\pi}{2}\frac{\partial}{\partial\rho}\left[x^{-\rho}F\left(\rho, \tfrac{1}{2}; 2; -\frac{1}{x}\right)\right]\Bigg|_{\rho=1} \\ &\sim \frac{\pi}{\sqrt{x}}\ln(4x) + \pi\ln x + \cdots \qquad x \ll 1. \end{aligned} \tag{14}$$

With the same considerations,

$$\frac{1}{2}\int_a^b \mathrm{d}t\ \sigma(t)\ u(t) \sim \tfrac{1}{2}N^2 - \alpha N \ln(2/\mathrm{e}^{1/2}) - \tfrac{1}{2}\alpha^2 \ln 4N + \tfrac{1}{2}Na - \tfrac{3}{2}\alpha\sqrt{Na} + \tfrac{1}{4}\alpha^2 \ln(4Na)\,. \tag{15}$$

Pooling together (11) and (15) we find by subtracting the very large terms

$$F(a,b) - F(a \ll b, b) \sim Na - 2\alpha\sqrt{Na} + \tfrac{1}{4}\alpha^2 \ln(4Na) \tag{16}$$

which gives

$$E_2(0,s) \sim \frac{\mathrm{e}^{-s/4+\alpha\sqrt{s}}}{s^{\alpha^2/4}} \tag{17}$$

upon scaling with respect to the hard-edge density, i.e. with the replacement $Na \to s/4$.

Before we proceed to give the result for the $\beta \neq 2$ case, we should like to mention that by *repeating the calculation in the far simpler situation where* $\alpha = 0$, *(8) can be solved trivially*, which gives $b - a = 4N$ (for $\beta = 2$) and the change in the free energy is equal to Na or $s/4$. This gives $E_2(0,s) = \mathrm{e}^{-s/4}$ [15, 16]. For $\alpha \neq 0$, $\beta = 2$, Tracy and Widom [9], found through an asymptotic expansion of a Painlevé V equation, higher-order terms, in $1/\sqrt{s}$, $1/s$ etc, and made a conjecture concerning the term independent of s†. It was observed in [9] that with $\alpha = \mp\frac{1}{2}$, the Bessel kernel reduces to the kernels which arise by scaling into the bulk of the spectrum of the Gaussian orthogonal and symplectic ensembles, respectively, provided one makes the replacement $s \to \pi^2 t^2$. It is interesting to see that by approaching the GOE and GSE through the 'back door', i.e. using the mapping of Tracy and Widom [9], the Coulomb-fluid approach, when applied to the Laguerre ensemble, supplies rather precise information‡.

3. Correction to the free energy when $\beta \neq 2$

This is simply found by adding the finite-temperature contribution

$$\frac{\delta F(a,b)}{\left(1 - \frac{1}{2}\beta\right)} = \int_a^b \mathrm{d}x \sigma(x) \ln[\sigma(x)] \sim -\ln\left[\frac{\beta\pi}{\mathrm{e}}\right] N - \frac{\alpha}{\beta}\ln N + \frac{\alpha}{2\beta}\ln 4Na \qquad a \ll b - a \tag{18}$$

to the free energy.

Collecting the appropriate terms together, we find that

$$-\ln E_\beta(0,s) \sim \frac{s}{2\beta} - \frac{2\alpha}{\beta}\sqrt{s} + \frac{\alpha^2}{2\beta}\ln s + \left(1 - \frac{\beta}{2}\right)\frac{\alpha}{2\beta}\ln s \tag{19}$$

which clearly reduces to (17) when $\beta = 2$§.

† The s-independent constant is $\ln \tau_\alpha$, where $\tau_\alpha = (2\pi)^{-\alpha/2} G(1+\alpha)$ and G is the Barnes G-function.

‡ The authors should like to thank Craig Tracy for bringing out this point. Specifically, $\ln E_2^{\alpha=\mp 1/2}(0, \pi^2 t^2) = \ln D_\pm(t) \sim -\pi^2 t^2/4 \mp \pi t/2 - \frac{1}{8}\ln(\pi t)$. The quantities $D_\pm(t)$ can be found in [17].

§ In order to facilitate comparison with [16], we make the unique identification; $S = 4s/\beta^2$. Equation (19) with $\alpha = 1$ agrees up to a proportional constant with the asymptotic expansion of (2.30) of [16]. Furthermore, (19) with with $\alpha = -\beta/2$, again agrees with the asymptotic expansion of equation (2.32) of [16]. In addition, by putting $\beta = 1$, $\alpha = -\frac{1}{2}$ in (19) we recover exactly the $\beta = 1$ value of (2.32) of [16], provided in all three cases the unique identification is made.

4. Conclusion

We should like to mention that by treating the constraint, i.e. (8), more accurately, it should be possible to produce the higher-order correction terms $1/\sqrt{s}$, $1/s, \ldots$ mentioned previously. However, the computation would become exceedingly complicated and clearly the methods of Tracy and Widom [9] have much to be desired.

The above calculations suggest, upon comparison with exact results, that the Coulomb-fluid approach is quite robust and may shed light on the level-spacing distribution of the q-Laguerre ensemble, which arises in the context of electronic transport in disordered systems†, with the potential

$$u(x, q) = \sum_{n=0}^{\infty} \ln\left[1 + (1-q)x\, q^n\right] \qquad q \in (0, 1) \tag{20}$$

which reduces to the ordinary Laguerre potential as $q \to 1^-$; $u(x, 1^-) = x$. We leave as a future project the determination of the level-spacing distribution with $u(x; q)$.

Note added. After this manuscript was completed we received a preprint on the level-spacing distribution of the Laguerre ensemble from Peter Forrester. We would like to thank the author for sending us the preprint. Equation (2.26a) of the preprint is a special case of (19) of our letter (α = integers) provided the identification of the parameters in footnote § (page 4) is made.

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† These types of ensembles are related to q-orthogonal polynomials associated with indeterminate classical moment problems, where the potential, $u(x) \sim [\ln x]^2$, for very large x, is marginally confining and has application to transport in disordered systems. See [18]. It can be shown that the level density at the origin, $\sigma_N(0, q)$, is $(1-q^N)/\ln(1/q)$.

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